Vertex-Disjoint Packing of Two Steiner Trees: Polyhedra and Branch-and-Cut

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Abstract. Consider the problem of routing the electrical connections among two large terminal sets in circuit layout. A realistic model for this problem is given by the vertex-disjoint packing of two Steiner trees (2VPST), which is known to be NP-complete. This work presents an investigation on the 2VPST polyhedra. The main idea is to depart from facet-defining inequalities for a vertex-weighted Steiner tree polyhedra. Some of these inequalities are proven to also define facets for the packing polyhedra, while others are lifted to derive new important families of inequalities, including proven facets. Separation algorithms are also presented. The resulting branch-and-cut code has an excellent performance and is capable of solving problems on grid graphs with up to 10000 vertices and 5000 terminals in a few minutes.

1 Introduction

The Vertex-disjoint Packing of Steiner Trees problem (VPST) is the following: given a connected graph $G = (V, E)$ and K disjoint vertex-sets $N_1, \ldots, N_K$, called *terminal sets*, find disjoint vertex-sets $T_1, \ldots, T_K$ such that, for each $k \in \{1, \ldots, K\}$, $N_k \subseteq T_k$ and $G[T_k]$ (the subgraph induced by T_k) is connected. Deciding whether a VPST instance is feasible or not is NP-complete, and many times a very challenging task. VPST has a wide application in layout of printed boards or chips, as it can model in an accurate way the routing of electrical connections among the circuit components (see Korte et al. [3] or Lengauer [4]).

This work studies the case where $K = 2$, the Vertex-Disjoint Packing of Two Steiner Trees (2VPST). This restricted problem is still NP-complete [3]. Although most VPST applications involves more than two terminal sets, many problems in circuit layout have only two large terminal sets, for instance the ground and the V_{cc} terminals. Finding a solution for these two large sets may represent a major step in solving the whole routing problem. In fact, it is usual in chip layout to route the ground and V_{cc} connections over special layers, separated from the remaining connections. This particular routing problem leads to the 2VPST.

We address an optimization version of the 2VPST: find two disjoint vertex-sets T_1 and T_2, such that $G[T_1]$ and $G[T_2]$ are connected, minimizing $|N_1 \setminus T_1| + |N_2 \setminus T_2|$. In other words, we look for a solution leaving the minimum possible number of unconnected terminals. It is also allowed to set weights in order to specify which terminals have priority in being connected. These features may be very useful for a practitioner and suggests an iterative approach for layout: if the 2VPST instance is not feasible, a best possible solution gives valuable information on how to make small changes in terminal placement to achieve feasibility.

A previous work that is somehow related to ours is Grötschel et al. [1, 2], where Edge-disjoint Packing of Steiner Trees (EPST) is studied. EPST gives an alternative model for the routing problem in circuit layout. Their branch-and-cut code could solve switchbox instances (G is a grid graph and all terminals are placed in the outer face) with up to 345 vertices, 24 terminal sets, and 68 terminals.

2 Formulation

For each vertex $i \in V$ define a binary variable x_i, where $x_i = 1$ if i belongs to T_1 and $x_i = 0$ if i belongs to T_2. This choice of variables obliges every vertex to belong either to T_1 or T_2. Of course, terminals of N_1 in T_2 and terminals of N_2 in T_1 are considered unconnected. A formulation for 2VPST follows.

$$
(\mathrm{F}) \left\{
\begin{array}{llll}
\text{Min } z = -\sum\limits_{i \in N_1} w_i + \sum\limits_{i \in V} w_i x_i & & & (1) \\
\text{s.t.} & \sum\limits_{i \in S} x_i - \sum\limits_{i \in R} x_i \leq 1 & \forall S \subseteq V,\ |S| = 2;\ \forall R \in \Delta(S) & (2) \\
 & -\sum\limits_{i \in S} x_i + \sum\limits_{i \in R} x_i \leq |R| - 1 & \forall S \subseteq V,\ |S| = 2;\ \forall R \in \Delta(S) & (3) \\
 & x_i \in \{0, 1\} & \forall i \in V & (4)
\end{array}
\right.
$$

Weight w_i must be negative if $i \in N_1$, positive if $i \in N_2$ and 0 otherwise. The minimum possible number of unconnected terminals matches z^* when $w_i = -1$ for all $i \in N_1$ and $w_i = 1$ for all $i \in N_2$. In any case the 2VPST instance is feasible iff $z^* = 0$. The symbol $\Delta(S)$ denotes the set of all minimal separators of S. A vertex-set R is a *separator of* S if any path joining the pair of vertices in S has at least one vertex in R. Constraints F.2 says: if both vertices in S are in T_1, then at least one vertex in any separator of S must also be in T_1. Constraints F.3 are complementary to F.2: if both vertices in S are in T_2, then at least one vertex in any separator of S must be in T_2.

3 Polyhedral Investigations

Many combinatorial optimization problems can be naturally decomposed into simpler subproblems. The 2VPST decompose into two Steiner-like subproblems. There are cases where a good polyhedral description for the subproblems yields a good description of the main polyhedra. On the other hand, sometimes even a complete subproblem description yields but a poor description of the main polyhedra. Experiences with column generation showed that this is indeed the case of 2VPST [6]. In these situations, there is a need to also look for *joint inequalities*, i.e. inequalities which are not valid for any subproblem polyhedra but valid for the main polyhedra.

Definition 1. *Given a graph $G = (V, E)$, a* v-tree of G *is a vertex-set $T \subseteq V$ such that $G[T]$ is connected. A* complementary v-tree of G *is a vertex-set $T \subseteq V$ such that $G[V \setminus T]$ is connected. A* double v-tree of G *is a v-tree T of G that is also a complementary v-tree of G. (G being clear, we say v-tree or double v-tree for short).*

Definition 2. *Let χ^U be the incidence vector of a vertex-set U. Define $P(G)$ as the convex hull of vectors χ^T such that T is a double v-tree of G. Define $P_1(G)$ as the convex hull of vectors χ^T such that T is a v-tree of G and $P_2(G)$ as the convex hull of vectors χ^T such that T is a complementary v-tree of G.*

The vectors defining polyhedra P correspond to the solutions of formulation F, so P is associated to 2VPST. Polyhedra P_1 and P_2 are associated to a Vertex Weighted Steiner Tree problem: given a graph with positive and negative vertex weights, find a tree with maximum/minimum total vertex weight. Note that P_1 and P_2 are complementary polyhedra and that every valid inequality for one can be converted into a valid inequality for the other by variable complementation, i.e. replacing x by $1 - x$. For instance, inequalities F.2 are valid for P_1 and the complementary inequalities F.3 are valid for P_2. Therefore whenever we talk about polyhedra P_1, it is implicit that similar results also hold for P_2.

Our idea is to use subproblem polyhedra P_1 and P_2 as a starting point for investigating main polyhedra P. We show that: (i) some inequalities that define facets of P_1 or P_2 also define facets of P and (ii) some of these inequalities which do not define facets of P can be lifted to derive new joint inequalities.

Theorem 1. *$P(G)$, $P_1(G)$ and $P_2(G)$ are full-dimensional.*

Proof. Let A be a spanning tree of G, rooted at an arbitrary vertex. For each $i \in V$, the vertex-set U_i containing i and his descendents in A is a double v-tree, so $\chi^{U_i} \in P(G)$. As $\chi^{\emptyset} \in P(G)$, there are $|V| + 1$ affine independent vectors in $P(G)$. As $P(G) \subseteq P_1(G) \cap P_2(G)$, the result follows. $\square$

3.1 The p-separator Facets of P_1

Definition 3. *Let S be a vertex-set, where $|S| \geq 2$. A vertex-set R is a* separator of S *if any path joining a pair of vertices in S has at least one vertex in R. A separator R of S is* proper *if $S \cap R = \emptyset$ and* minimal *if no other separator of S is contained in R. The symbol $\Delta(S)$ indicates the set of all proper and minimal separators of S. Given a separator R of S and a vertex $i \in V$, define S_i^R as the maximum subset of vertices in S that can belong to a v-tree T_i containing i and such that $T_i \cap R \subseteq \{i\}$. Note that a separator R of S is minimal iff $|S_i^R| \geq 2$ for each $i \in R$.*

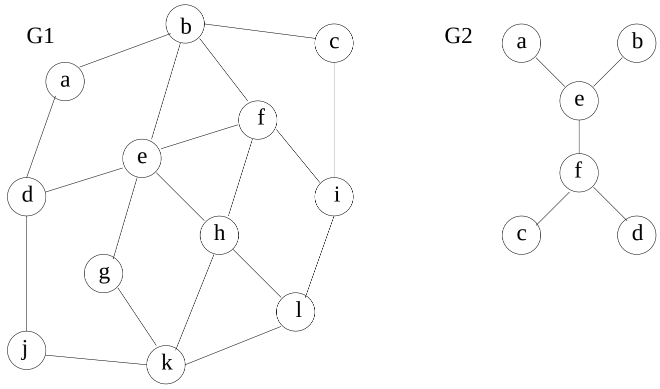

Fig. 1. Graphs for examples

For example, in graph $G1$ (Fig. 1) taking $S = \{a, c, h\}$, $R = \{b, e, k, i\}$ is a proper minimal separator of S, $S_b^R = \{a, c, h\}$, $S_e^R = \{a, h\}$, $S_j^R = \{a\}$ and $S_g^R = \emptyset$. The following theorem suggests that the concept of separator is fundamental to understand the facial structure of P_1. The results of this subsection are proved in the appendix.

Theorem 2. *Every facet of $P_1(G)$ can be defined by a trivial inequality ($-x_i \leq 0$ or $x_i \leq 1$) or by an inequality in format $\sum_{i \in S} \beta_i x_i - \sum_{i \in R} \alpha_i x_i \leq 1$, where β and α are vectors of positive numbers and R is a proper separator of S.*

We define a family of potentially facet-defining inequalities.

Definition 4. *Let S be a vertex-set, where $|S| = p \geq 2$, and $R \in \Delta(S)$. The following inequalities are called* p-separators, *where α_i is an integer greater than $|S_i^R| - 1$.*

$$\sum_{i \in S} x_i - \sum_{i \in R} \alpha_i x_i \leq 1$$

Constraints F.2 are 2-separators. In graph $G1$, $x_a + x_f - x_b - x_e - x_j \leq 1$ is an example of such inequality. In the same graph, $x_a + x_f + x_g - x_b - 2x_e - 2x_k \leq 1$ is a 3-separator valid for $P_1(G1)$. Sometimes taking $\alpha_i = |S_i^R| - 1$ for all $i \in R$ does not yield a valid inequality for P_1. For instance, in graph $G2$ (Fig. 1) taking $S = \{a, b, c, d\}$ and $R = \{e, f\}$, the 4-separator $\sum_{i \in S} x_i - x_e - x_f \leq 1$ is not valid for $P_1(G2)$. But 4-separators $\sum_{i \in S} x_i - 2x_e - x_f \leq 1$ and $\sum_{i \in S} -x_e - 2x_f \leq 1$ are valid and facet-defining.

Theorem 3. *For each S and each $R \in \Delta(S)$, there is at least one p-separator inequality which is valid and facet-defining for $P_1(G)$.*

Corollary 1. *If the p-separator $\sum_{i \in S} x_i - \sum_{i \in R}(|S_i^R| - 1)x_i \leq 1$ is valid for $P_1(G)$, then it defines a facet of $P_1(G)$.*

The above theorem does not tell how to calculate the α coefficients of a p-separator facet. But given sets S and $R \in \Delta(S)$, the coefficients of a p-separator valid for P_1 can be computed by the following procedure:

Procedure Compute α
$M \longleftarrow \emptyset$;
For each $i \in R$ {
If (exists $j \in M$ such that $S_i^R \cap S_j^R = \emptyset$)
Then { $\alpha_i \longleftarrow |S_i^R|$; }
Else { $\alpha_i \longleftarrow |S_i^R| - 1$; $M \longleftarrow M \cup \{i\}$; }
}

As example, take graph G2 with $S = \{a, b, c, d\}$ and $R = \{e, f\}$. If we pick vertices in order (e, f), the procedure yields $\alpha_e = 1$ and $\alpha_f = 2$. If the order is reversed, it yields $\alpha_f = 1$ and $\alpha_e = 2$. Note that if $p \leq 3$, the procedure always yields $\alpha_i = |S_i^R| - 1$ for all $i \in R$. Then, by Corollary 1, all such 2-separators and 3-separators define facets of $P_1(G)$. The above procedure is not optimal, if $p \geq 4$ the resulting p-separators are not always facet-defining.

3.2 From P_1 to P: p-separators, Lifted p-separators and p-crosses

If a certain p-separator defines a facet of $P_1(G)$, it may also define a facet of $P(G)$, so it is interesting to investigate when it happens. Also interesting is to investigate when this does not happen: in this case there is room for lifting the p-separator, producing a new joint inequality.

Definition 5. *The inequalities valid for $P(G)$ that can be obtained by lifting α coefficients of a p-separator facet of $P_1(G)$ or $P_2(G)$ are called* lifted p-separators.

The following definitions and lemma (proved in the appendix) are necessary for announcing and proving the main results of this section.

Definition 6. *Let T be a double v-tree of G and t a vertex in T. A* sequence of nested double v-trees from t to T *is any sequence of double v-trees $T_1, \ldots, T_{|T|}$ such that $\{t\} = T_1 \subset \ldots \subset T_{|T|} = T$. (In other words, for each i, $1 < i \leq |T|$, T_{i-1} may be obtained from T_i by removing a vertex distinct from t).*

Lemma 1. *Suppose G is biconnected. Given a double v-tree T of G and $t \in T$, it is always possible to construct a sequence of nested double v-trees from t to T.*

Definition 7. *A separator R of $S = \{s_1, \ldots, s_p\}$* induces *the following* structure *in G. Graph $G[V \setminus R]$ is split in a number of connected components: those containing exactly one vertex in S are called* A-components *and those with no vertices in S are called* B-components. *Define A_i as the vertex-set of the A-component containing s_i (when R is not proper some A-components do not exist). The vertex-sets corresponding to B-components are numbered arbitrarily as $B_1, \ldots, B_q$. Note that $|S_i^R| = 0$ if i belongs to a B-component and $|S_i^R| = 1$ if i belongs to a A-component.*

Theorem 4. *Suppose G is biconnected. Let $\sum_{i\in S} x_i - \sum_{i\in R}(|S_i^R|-1)x_i \leq 1$ be a valid p-separator, where R induces a structure without B-components and that every A-component is a double v-tree. If for every $i \in R$, there is a double v-tree U_i containing $\{i\} \cup S_i^R$ such that $U_i \cap R = \{i\}$, this inequality defines a facet of $P(G)$.*

Proof. Let $a^t x \leq a_0$ be a valid inequality for $P(G)$ such that

$$F = \{x \in P(G) \,|\, \sum_{i\in S} x_i - \sum_{i\in R}(|S_i^R|-1)x_i = 1\} \subseteq F_a = \{x \in P(G) \,|\, a^t x = a_0\} \ .$$

We show that $a^t x \leq a_0$ is an scalar multiple of this p-separator. Consider the structure induced by R. For every A-component A_j, applying Lemma 1, construct a sequence of nested double v-trees from s_j to A_j. For each double v-tree T in this sequence, $\chi^T \in F$. So $a_{s_j} = a_0$ and for every $i \neq s_j \in A_j$, $a_i = 0$. For every vertex $i \in R$, $\chi^{U_i} \in F$, where U_i is the double v-tree given by hypothesis. So $a_i = -(|S_i^R|-1)a_0$. □

For example, in graph G1, the 2-separator $x_a + x_f - x_b - x_e - x_j \leq 1$, 3-separator $x_a + x_f + x_g - x_b - 2x_e - 2x_k \leq 1$ and 4-separator $x_a + x_c + x_g + x_h - 2x_b - 2x_e - x_i - 2x_k \leq 1$ define facets of $P(G1)$. The converse of Theorem 4 does not hold even if $p = 3$, so we need some additional conditions to be able to lift α coefficients.

Theorem 5. *Suppose G is biconnected. Let $\sum_{i\in S} x_i - \sum_{i\in R}(|S_i^R|-1)x_i \leq 1$ be a valid p-separator, where R induces a structure without B-components and that every A-component is a double v-tree. Suppose that there is a vertex $i \in R$ not satisfying Theorem 4 conditions. If $|S_i^R| = p$, the coefficient α_i can be lifted in at least one unit.*

Proof. We show that for every double v-tree T containing i, $\gamma_T = |S \cap T| - \sum_{i\in R\cap T} \alpha_i \leq 0$. If T contains S_i^R, $|S \cap T| = |S_i^R|$. As T must contain another vertex j of R, $\gamma_T \leq |S_i^R| - (|S_i^R|-1) - (|S_j^R|-1) = 2 - |S_j^R| \leq 0$. If T does not contain S_i^R, $\gamma_T \leq (|S_i^R|-1) - (|S_i^R|-1) \leq 0$. □

If $p = 2$, the above results can be collected as follows:

Corollary 2. *Suppose G is biconnected. Let $\sum_{i\in S} x_i - \sum_{i\in R} x_i \leq 1$ be a 2-separator, where R induces a structure without B-components. This inequality defines a facet of $P(G)$ iff for every $i \in R$, there is a double v-tree U_i containing $\{i, s_1, s_2\}$ such that $U_i \cap R = \{i\}$*

Proof. As R is minimal, every vertex in R is adjacent to both A_1 and A_2. So A_1 and A_2 are double v-trees. So the "if" part follows from Theorem 4. For each $i \in R$, $|S_i^R| = 2$. So the "only if" part follows from Theorem 5. □

For examples, in graph G1, the 2-separator $x_a + x_c - x_c - x_f - x_e - x_h - x_l \leq 1$ do not define a facet of $P(G1)$, by Corollary 2, α_e or α_h can be lifted to 0. The

3-separator $x_a + x_c + x_g - x_b - 2x_e - 2x_k \leq 1$ also do not define a facet $P(G1)$, by Theorem 5, α_e can be lifted to 1.

In graph G3 (Fig. 2), the 2-separator $x_a + x_b - x_d - x_e - x_f \leq 1$ is not facet-defining, although it satisfies the conditions of Corollary 2, except that R induces a structure with B-components. This 2-separator is dominated by the lifted 3-separator $x_a+x_b+x_c-x_d-x_e-x_f \leq 1$ (from 3-separator $x_a+x_b+x_c-2x_d-2x_e-2x_f \leq 1$). In graph G4 (Fig. 2), 3-separator $x_a + x_c + x_e - x_b - x_d - x_f - x_h \leq 1$ is not facet-defining, although it satisfies the conditions of Theorem 4, except that the A-component formed by $\{c, g\}$ is not a double v-tree. This 3-separator is dominated by $x_a + x_c + x_g + x_e - x_b - x_d - x_f - x_h \leq 1$.

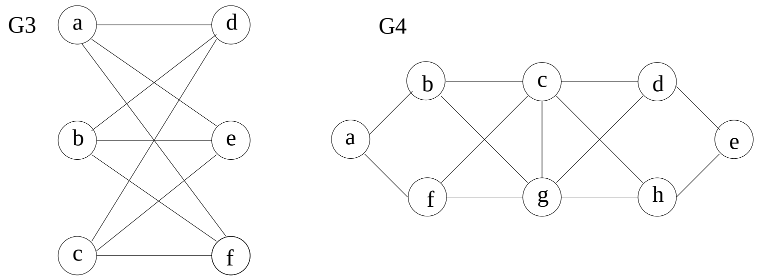

Fig. 2. Graphs for examples

The concept of lifted p-separators acts as a convenient classifying device, permitting us to view lots of joint inequalities in a single framework. Even though Theorem 5 does not cover all cases when a p-separator facet of $P_1(G)$ can be lifted. It is certainly possible to improve these results, but difficulties appear. For instance, when $p \geq 4$ it is not clear even how to calculate the original α coefficients of a p-separator facet of $P_1(G)$. Also, it is usual that several such coefficients can be lifted at once. Conditions for this "multilifting" are more complicated. Most of all, it is cumbersome to design separation procedures for lifted p-separators departing from the original p-separator.

We give a direct characterization of an important subfamily of lifted p-separators that permitted us to overcome such difficulties. These inequalities turned out to be essential to good branch-and-cut performance.

Definition 8. *Let $S = \{s_1, \ldots, s_p\}$ and $R = \{r_1, \ldots, r_p\}$ be two disjoint vertex-sets with $p \geq 2$. Suppose that for any integers a, b, c, d in the range $\{1, \ldots, p\}$, with $a \leq b < c \leq d$, there are no two vertex-disjoint paths joining s_a to s_c and r_b to r_d (in other words, the vertices in every path from r_b to r_d define a separator of $\{s_a, s_c\}$ and vice-versa). Then the following inequality is called a p-*cross

$$\sum_{i \in S} x_i - \sum_{i \in R} x_i \leq 1$$

In graph G1, $x_e + x_j - x_d - x_g \leq 1$ is a 2-cross and $x_a + x_c + x_l + x_j - x_b - x_i - x_k - x_d \leq 1$ is a 4-cross. Most p-crosses are lifted p-separators, it suffices to "complete" R until a proper minimal separator of S is found. For instance, 3-cross $x_a+x_c+x_k-x_b-x_l-x_d \leq 1$ is a lifting of 3-separator $x_a+x_c+x_k-2x_b-x_f-x_l-x_d \leq 1$ facet of $P_1(G1)$ or $-x_b-x_l-x_d+x_a+x_c+2x_e+x_f+x_k \leq 4$ facet of $P_2(G1)$.

Theorem 6. *The p-crosses are valid for P.*

Proof. By induction on p. The 2-crosses are clearly valid. Suppose that $\sum_{i=1}^{p} x_{s_i} - \sum_{i=1}^{p} x_{r_i} \leq 1$ is a p-cross with $p \geq 3$. By induction hypothesis, $(p-1)$-crosses $\sum_{i=1}^{p-1} x_{s_i} - \sum_{i=1}^{p-1} x_{r_i} \leq 1$ and $\sum_{i=2}^{p} x_{s_i} - \sum_{i=2}^{p} x_{r_i} \leq 1$ are valid. Summing these inequalities with 2-cross $x_{s_1} + x_{s_p} - x_{r_1} - x_{r_p} \leq 1$, dividing the result by 2 and rounding down the right-hand-side, the desired p-cross is obtained. □

Theorem 7. *If G is biconnected, every 2-cross defines a facet of $P(G)$.*

Proof. Let $x_{s_1} + x_{s_2} - x_{r_1} - x_{r_2} \leq 1$ be a 2-cross and $a^t x \leq a_0$ be a valid inequality for $P(G)$ such that

$$F = \{x \in P(G) \,|\, x_{s_1} + x_{s_2} - x_{r_1} - x_{r_2} = 1\} \subseteq F_a = \{x \in P(G) \,|\, a^t x = a_0\} \ .$$

We show that $a^t x \leq a_0$ is an scalar multiple of this 2-cross. As G is biconnected, there are two vertex-disjoint paths, except in the endpoints, joining r_1 to r_2. Let I_1 and I_2 be the set of vertices of any two such paths. As I_1 and I_2 are separators of S, it is not possible to have $S \subseteq I_1$ or $S \subseteq I_2$.

Case (a): I_1 or I_2 is a proper separator of S.

Suppose I_1 is a proper separator of S and consider the induced structure. As G is biconnected, each component is adjacent to at least two vertices in I_1. We assert that at least one of the two A-components is adjacent to both r_1 and r_2. If this is not true, the path corresponding to I_2 must pass through a B-component and I_1 would not be a separator of S. Suppose A_2 is an A-component adjacent to r_1 and r_2.

Subcase (a.1): A_1 is adjacent to a vertex in $I_1' = I_1 \setminus \{r_1, r_2\}$. In this case, $C_1 = A_1 \cup I_1' \cup \{B_k \,|\, B_k \text{ is adjacent to } I_1'\}$ and $C_2 = A_2$ are double v-trees. Construct a sequence of nested double v-trees from s_1 to C_1 and from s_2 to C_2. For each T in these sequences, $\chi^T \in F$. So $a_{s_1} = a_{s_2} = a_0$ and $a_i = 0$ if $i \in (C_1 \cup C_2) \setminus S$. As $\chi^{C_1 \cup C_2 \cup \{r_1\}}$ and $\chi^{C_1 \cup C_2 \cup \{r_2\}} \in F$, $a_{r_1} = a_{r_2} = -a_0$. Each component B_k not adjacent to I_1' must be adjacent to both r_1 and r_2. Let j be a vertex in B_k adjacent to r_1. Construct a sequence of nested double v-trees from j to B_k. For each T in this sequence $U_T = C_1 \cup C_2 \cup \{r_1\} \cup T$ is also a double v-tree and $\chi^{U_T} \in F$. So $a_i = 0$, for all $i \in B_k$.

Subcase (a.2): A_1 is not adjacent to I_1'. In this case, A_1 is adjacent to r_1 and r_2. Let $C_1 = A_1$, $C_2 = A_2$ and proceed similar to case a.1, concluding that $a_{s_1} = a_{s_2} = a_0$, $a_i = 0$ if $i \in (C_1 \cup C_2) \setminus S$ and $a_{r_1} = a_{r_2} = -a_0$. Now define C_3 as $I_1' \cup \{B_k \,|\, B_k \text{ is adjacent to } I_1'\}$. Let j be a vertex in I_1' adjacent to r_1. Construct a sequence of nested double v-trees from j to C_3. For each T in this sequence

$U_T = C_1 \cup C_2 \cup \{r_1\} \cup T$ is also a double v-tree and $\chi^{U_T} \in F$. So $a_i = 0$, for each $i \in C_3$. B-components adjacent to both r_1 and r_2 are treated in a similar way.

Case (b): I_1 and I_2 are not proper separators of S.

Suppose I_1 contains s_1 and I_2 contains s_2. Consider the structure induced by separator I_1. As I_2 contains s_2, A_2 is adjacent to r_1 and r_2. Let $C_1 = I_1' \cup \{B_k \mid B_k \text{ is adjacent to } I_1'\}$, $C_2 = A_2$ and proceed similar to subcase a.1. B-components adjacent to both r_1 and r_2 are also treated like in a.1. □

Using similar techniques it is possible to prove that under mild conditions p-crosses with $p \geq 3$ also define facets of $P(G)$. For instance, supposing G is biconnected and planar, all p-crosses are facet-defining.

4 Separation Procedures

The 2-separators can be separated by applying the min-cut algorithm over a suitable network. When $p \geq 3$, the procedures to find violated p-separators inequalities are harder to devise. One heuristic procedure to find such a p-separator is by the successive application of the 2-separator procedure. Note that if $p \geq 4$ there is the additional complication of calculating proper α coefficients.

The p-crosses can be separated in polynomial time over any graph G, for fixed p. This follows from definition 8, since there exists an $O(|V|.|E|)$ algorithm to determine if it is possible to connect two given pairs of vertices by vertex-disjoint paths (Shiloach [7]). However, far better p-cross separation schemes may exist if the particular topological structure of G is considered. If G is planar and supposing that no 2-separator is violated, a violated p-cross only occurs when the sequence $s_1, r_1, \ldots, s_p, r_p$ happen in this cyclic order on some face f of G. By this property, p-cross inequalities can be separated in $O(|V|)$ over planar graphs, by dynamic programming.

5 Computational Results

The branch-and-cut for 2VPST was implemented in C++ using the ABACUS framework [8] and CPLEX 3.0 LP solver, running on a Sun ULTRA workstation. The code used 2-separators, 3-separators (heuristically separated) and p-crosses.

In all experiments described in table 1, G is a square grid graph (grid graphs are typical in circuit layout applications). Two kind of instances were generated:

- Unweighted instances with $|N_1| = |N_2| = |V|/4$. The terminals in each of these sets are randomly chosen among the vertices of G. Vertices in N_1 have weight -1 and vertices in N_2 have weight 1. These instances are referenced as u_n_$Inst$, where $n = |V|$ and $Inst$ is the instance number. The value z^* gives the number of unconnected terminals in an optimal solution.
- Weighted instances. For each unweighted instance we generated a corresponding weighted instance, by picking an integer w_i from an uniform distribution in the range $[-10, -1]$ if $i \in N_1$ and from $[1, 10]$ if $i \in N_2$. The

value z^* gives the sum of the absolute weights of the unconnected terminals in an optimal solution. Weighted instances are referenced as w_n_$Inst$.

Table 1. Branch-and-Cut Results

Instance	LB_r	z^*	Nd	LPs	T	Instance	LB_r	z^*	Nd	LPs	T
u1600_1	45.7	46	4	96	20	w1600_1	222.0	222	2	35	9
u1600_2	40.0	40	2	40	8	w1600_2	218.0	218	1	25	5
u2500_1	46.0	46	2	56	33	w2500_1	233.0	233	2	37	14
u2500_2	55.0	55	3	49	32	w2500_2	227.5	228	3	39	17
u3600_1	76.5	77	2	45	41	w3600_1	364.0	364	2	48	33
u3600_2	74.0	74	4	57	50	w3600_2	311.0	311	2	74	54
u4900_1	99.1	100	4	116	170	w4900_1	420.0	420	2	77	130
u4900_2	92.0	92	3	66	108	w4900_2	400.0	400	3	53	90
u6400_1	104.0	104	3	126	443	w6400_1	436.0	436	3	106	295
u6400_2	127.0	128	4	123	322	w6400_2	534.0	534	3	128	367
u8100_1	131.5	132	4	217	1207	w8100_1	549.5	550	4	113	563
u8100_2	115.0	115	6	125	713	w8100_2	467.0	467	3	82	264
u10000_1	159.8	161	8	120	676	w10000_1	685.0	685	3	60	345
u10000_2	151.5	152	4	157	991	w10000_2	632.5	634	7	116	621

In Table 1, the $\mathbf{LB}_r$ column presents the lower bound obtained in the root node, $\mathbf{z}^*$ is the value of the optimal integer solution, **Nd** is the number of nodes explored in the branch-and-bound tree, **LPs** is the total number of solved LPs and **T** is the total cpu time in seconds spent by the algorithm. Remark that LB_r bounds includes p-crosses and 3-separators. The lower bounds given by the linear relaxation of formulation F alone are rather weak, for example, in instance u1600_1 it is 8.7. Figure 3 shows an optimal solution of instance u1600_1, the black boxes represent the terminals in N_1, the empty boxes are the terminals in N_2, the full lines are edges of a spanning tree for $G[T_1]$ and the dotted lines a spanning tree for $G[T_2]$.

The code turned out to be robust under changes on terminal placement, terminal density and weight distribution. In fact, we used the same set of parameters for solving all instances in table 1, weighted and unweighted. An interesting experience is to generate a random terminal placement avoiding the most obvious causes of infeasibility: terminals on the outer face, two pairs of opposite terminals crossed over the same grid square and terminals blocked by its neighbors. An optimal solution of such an instance (called t1600_1) with 800 terminals and unitary weights is shown in Fig. 4. The unconnected terminals are in coordinates (4,33), (8,21), (33,36) and (34,15). This solution was obtained in 21 seconds.

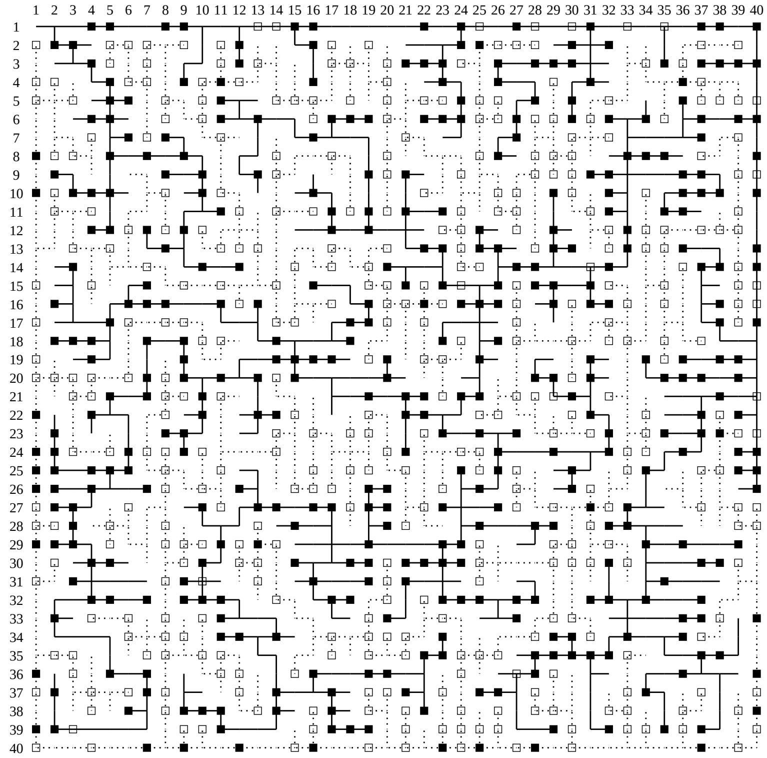

Fig. 3. An optimal solution of instance u1600_1, $z^* = 46$.

Appendix

Proof (Theorem 2). Suppose that a facet F of $P_1(G)$ is defined by $a^t x \le a_0$. The right-hand-side a_0 must be non-negative, otherwise this inequality would not be valid for the empty v-tree. So, w.l.o.g. we can suppose that $a_0 = 0$ or $a_0 = 1$. Let S be the vertex-set corresponding to positive coefficients in a and R the vertex-set corresponding to negative coefficients.

Case (a): $a_0 = 0$. The set S must be empty, otherwise $a^t x \le 0$ would not be valid for the v-tree formed by a single vertex in S. The set R must be a singleton $\{u\}$, since if there are other vertices in R, $a^t x \le 0$ would be dominated by $-x_u \le 0$. So F can be defined by this trivial inequality.

Case (b): $a_0 = 1$. S can not be empty. If S is a singleton $\{u\}$, F can be defined by the trivial inequality $x_u \le 1$. Suppose $|S| \ge 2$. Graph $G[V \setminus R]$ is split into a number of connected components. No two vertices in S can belong to the same such component, since if u and w are two vertices in S in the same component then $a^t x \le 1$ would be a linear combination of valid inequalities $\sum_{i \in S \setminus \{u,w\}} a_i x_i + (a_u + a_w) x_u + \sum_{i \in R} a_i x_i \le 1$ and $\sum_{i \in S \setminus \{u,w\}} a_i x_i + (a_u + a_w) x_w + \sum_{i \in R} a_i x_i \le 1$. Therefore R is a proper separator of S. $\square$

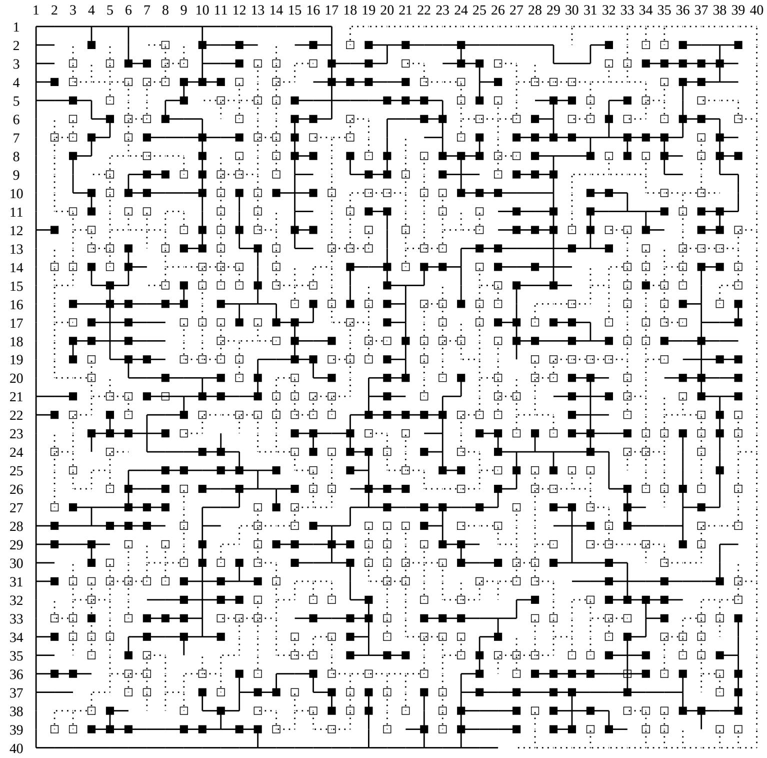

Fig. 4. An optimal solution of instance t1600_1, $z^* = 4$.

The following lemma and definition are used in the proof of Theorem 3.

Lemma 2. *(Sequential Maximum Lifting Lemma, pages 261-263 of [5]) Let P be a polyhedron in $\mathbb{R}^n$. Let N be the set of indices $\{1, \ldots, n\}$ and Z_0 a subset of N. Define P^{Z_0} as the polyhedron $P \cap \{x \in \mathbb{R}^n | x_i = 0, i \in Z_0\}$. Suppose $\sum_{i \in N \setminus Z_0} a_i x_i \leq a_0$ is a valid inequality for P^{Z_0} defining a face of dimension d. Let $\sum_{i \in N \setminus Z_0} a_i x_i + \sum_{i \in Z_0} m_i x_i \leq a_0$ be any inequality obtained by a sequential maximum lifting of one coefficient in Z_0 at a time (in any order). This lifted inequality defines a face of P of dimension $d + |Z_0|$, and if the original inequality defines a facet of P^{Z_0}, the lifted inequality defines a facet of P.*

Definition 9. *Let T be a v-tree of G and t a vertex in T. A* sequence of nested v-trees from t to T *is any sequence of v-trees $T_1, \ldots, T_{|T|}$ such that $\{t\} = T_1 \subset \ldots \subset T_{|T|} = T$. Note that given v-tree T and t, is always possible to construct such a sequence.*

Proof (Theorem 3). Consider the structure induced by R (see definition 7). Inequality $\sum_{i \in S} x_i \leq 1$ is valid for $P_1(G)^{Z_0}$, where $Z_0 = R \cup (\cup_{j=1}^{q} B_j)$. Define $F = \{x \in P_1(G)^{Z_0} | \sum_{i \in S} x_i = 1\}$. For each $j \in \{1, \ldots, p\}$, A_j is a v-tree.

Construct a sequence of nested v-trees from s_j to A_j. For every T in these sequences, $\chi^T \in F$. So there are $\sum_{j=1}^{p} |A_j|$ affine independent vectors in F and $\sum_{i \in S} x_i \leq 1$ defines a facet of $P_1(G)^{Z_0}$.

Number the vertices of R from r_1 to $r_{|R|}$. Define $R_0 = \emptyset$ and R_j as the set $\{r_1, \ldots, r_j\}$. For each $k \in \{1, \ldots, q\}$, define $f(k)$ as the smaller number of a vertex in R adjacent to B-component B_k. Define $W_j = R_j \cup \{B_k | f(k) \leq j\}$ and $Z_j = Z_0 \setminus W_j$. We show that for each $j \in \{0, \ldots, |R|\}$, there is an inequality $\sum_{i \in S} x_i - \sum_{i \in R_j} \alpha_i x_i \leq 1$, where each α_i is integer and satisfies $\alpha_i \geq |S_i^R| - 1$, defining a facet of $P_1(G)^{Z_j}$.

This is true when $j = 0$. Suppose $1 \leq j \leq |R|$. Pick the inequality corresponding to $j - 1$. First lift the coefficient of r_j. As there exists a v-tree T_j such that $T_j \cap R = \{r_j\}$, $T_j \cap S = S_j^R$ and $T_j \cap (\cup_{j=1}^{q} B_j) = \emptyset$, $\alpha_j \geq |S_j^R| - 1$. As the lifting is maximum, α_j is integral and there exists a v-tree $U \subseteq (\cup_{j=1}^{p} A_j \cup W_{j-1} \cup \{r_j\})$ such that $r_j \in U$ and $|S \cap U| - \sum_{i \in R_j \cap U} \alpha_i = 1$. For each B-component B_k such that $f(k) = j$, take a vertex $t \in B_k$ adjacent to r_j. Construct a sequence of nested v-trees from t to B_k. Make a maximum lifting of the coefficients of vertices in B_k in the same order this vertices are introduced in the sequence. Since $U \cup T$ is a v-tree for every T in the sequence, all such coefficients must be zero. Repeating this procedure until $j = |R|$, a p-separator defining a facet of $P_1(G)$ is obtained. $\square$

Proof (Correctness of procedure Calculate α). Let U be a v-tree and let $Q = R \cap U$. We must show that $|S \cap U| - \sum_{i \in Q} \alpha_i \leq 1$. Assume that Q is not empty, otherwise this inequality is clearly true. Sort the vertices in Q as $q_1, \ldots, q_{|Q|}$, using the same order in which these vertices were considered in procedure Calculate α. Define Q_i as the set $\{q_1, \ldots, q_i\}$ and $S_{Q_i}^R$ as $\cup_{j \in Q_i} S_j^R$.

We assert that for each $i \in \{1, \ldots, |Q|\}$, $\gamma_i = |S_{Q_i}^R \cap U| - \sum_{i \in Q_i} \alpha_i \leq 1$. This is true when $i = 1$. Suppose $1 < i \leq |Q|$. If $\alpha_i = |S_i^R|$, then $\gamma_i \leq \gamma_{i-1} \leq 1$. Suppose $\alpha_i = |S_i^R| - 1$. The procedure assures that if there exists a vertex $j \in Q_{i-1}$ with $\alpha_j = |S_j^R| - 1$ then $S_i^R \cap S_j^R \neq \emptyset$, and $\gamma_i \leq \gamma_{i-1} \leq 1$. If there is no such vertex $\gamma_{i-1} \leq 0$, so $\gamma_i \leq 1$. As $|S \cap U| - \sum_{i \in Q} \alpha_i \leq \gamma_{|Q|} \leq 1$, the result follows. $\square$

Proof (Lemma 1). If $T = \{t\}$ the result is immediate. Otherwise, suppose that a partial sequence $T = T_{|T|} \supset \ldots \supset T_i$, $|T| \geq i > 1$, is already formed. Vertices in double v-tree T_i, different from t, but adjacent to a vertex in $V \setminus T_i$ will be called *candidate vertices* (if $T = V$, all vertices in $V - t$ are candidates). As G is biconnected, there is at least one candidate vertex. We show that is always possible to choose a candidate to be removed from T_i to form double v-tree T_{i-1}.

Let v_0 be a candidate. If $T_i - v_0$ is a v-tree, v_0 is chosen. Otherwise $G[T_i - v_0]$ is separated into $j \geq 2$ connected components with vertex-sets $C_1^1, \ldots, C_j^1$. As T_i is a v-tree, at least one vertex in C_j^1 is adjacent to v_0. As G is biconnected, at least one vertex in each C_j^1 is adjacent to $V \setminus T_i$. Adjust notation for C_1^1 to be a set not containing t. Let v_1 be a candidate vertex in C_1^1. $G[C_1^1 - v_1]$ is separated into $j \geq 0$ connected components with vertex-sets $C_1^2, \ldots, C_j^2$. As C_1^1 is a v-tree, at least one vertex in each C_j^2 is adjacent to v_1. As G is biconnected, at least one vertex in each C_j^2 is adjacent to v_0 or to $V \setminus T_i$. If at least one vertex in each C_j^2 is adjacent to v_0 (or $j = 0$), v_1 is chosen. Otherwise, adjust notation for C_1^2 to be a set containing a candidate vertex.

By repeating this procedure, we get a sequence $v_1, \ldots, v_k$ of candidate vertices contained in smaller sets C_1^k. If each component of $G[C_1^k - v_k]$ is adjacent to $T_i \setminus C_1^k$, v_k is the chosen candidate and $T_i - v_k$ is a v-tree. In the worst case, the procedure is repeated until $C_1^k = \{v_k\}$. $\square$

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